

Categories with Distinguished Objects

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PART I. CDO: Categories with Distinguished Objects

Sol Feferman: What rests on what?

VP: Cycles are ok.

ZFC starts from the binary relation \in of **set membership**,
CDO starts from the “binary” operation \circ of **function composition**,
in both cases satisfying certain properties.

(For CDO, “binary” should be understood formally not as $\{0, 1\}$ but as $\{0 \leq 1\}$; informally, as a partial binary operation.)

~~No colimits, functors, natural transformations, adjunctions, monads, etc.~~

Any category *without* distinguished objects in the CDO framework is inconsistent in the sense that all its objects are empty.

Categories of sets and functions

Definition 1. A **category of sets and functions** is a category C with a rigid distinguished object $\mathbf{1}$ (the primitive set) satisfying the following axiom.

Axiom A1. Equivalent functions are equal.

Terminology: rigid, sets, elements, functions, actions, equivalence:

- An object is **rigid** when it only has one endomorphism.
- A **set** X is any object of C .
- An **element** $x \in X$ is a morphism $x : \mathbf{1} \rightarrow X$.
- A **function** $f : X \rightarrow Y$ is any morphism $f : X \rightarrow Y$.
- The **action** of f on an element $x \in X$ is the composite $fx \in Y$.
- Two functions $f, g : X \rightarrow Y$ are **equivalent** when they have the same action on all elements of X .

The Category **Set** of all sets and functions

Two modes of extension of any category of sets:

1. By morphisms, preserving sets (can't add morphisms from **1**) and subject to **A1** (can't add morphisms whose actions duplicate those of an extant morphism).
2. By sets, subject to extension mode 1. (Elements fixed "at birth".)

A category of sets is **complete** when it is equivalent to its every extension.

Axiom A2. (Existence and uniqueness) There exists a complete category of sets, and all complete categories of sets are equivalent.

Definition 2. **Set** is a complete category of sets and functions,

Set is only defined up to equivalence.

A category C of graphs and graph homomorphisms is a CDO with two distinguished objects \mathbf{V} (the primitive no-edge graph) such that there are two morphisms (s, t) from \mathbf{V} to \mathbf{E} .

A **graph** G (**homomorphism** $h : G \rightarrow G'$) is an object (morphism) of C .

A **vertex** v (**edge** e) is a morphism $v : \mathbf{V} \rightarrow G$ ($e : \mathbf{E} \rightarrow G$).

Edge e has

- A **source vertex** $es : \mathbf{V} \xrightarrow{s} \mathbf{E} \xrightarrow{e} G$ (contravariant action of s)
- A **target vertex** $et : \mathbf{V} \xrightarrow{t} \mathbf{E} \xrightarrow{e} G$

Graph homomorphisms respect source and target as a consequence of associativity of composition: $h(es) = (he)s$ where $hes : \mathbf{V} \xrightarrow{s} \mathbf{E} \xrightarrow{h} G'$. (Conventionally notated as $h(s(e)) = s(h(e))$).

Theories

A **theory** T is a small category.

Set theory: $\mathbf{1}$ (or just \bullet) (simpler than ZFC set theory.)

Graph theory: $\mathbf{V} \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{t} \end{array} \mathbf{E}$ (note opposite)

The theory T of a CDO C is the opposite J^{op} of the **base** or full subcategory J of C whose objects are the distinguished objects. $T = J^{op}$.

Opposite because the morphisms of J act contravariantly on elements.

Each object of T is understood as a **sort**, e.g. $\mathbf{1}$, \mathbf{V} , \mathbf{E} .

Each morphism of T is understood as an **operation symbol**, e.g. $s(e)$, $t(e)$.

Action contravariantly interprets operation symbols as operations on edges.

Extensions of graph theory

Reflexive graphs: Expands the base J for graphs with three distinct morphisms $i : \mathbf{E} \rightarrow \mathbf{V}$, $si : \mathbf{E} \rightarrow \mathbf{E}$, $ti : \mathbf{E} \rightarrow \mathbf{E}$.

This can be understood as the category of the positive finite ordinals ≤ 2 and their monotone functions.

By Morita equivalence omitting “positive” ($\{0, 1, 2\}$) makes no difference.

Simplicial sets: Omit “ ≤ 2 ” (all finite ordinals Δ). $T = \Delta^{op}$.

Cubical sets: $T = \mathbf{Fbip}$. \mathbf{Fbip} is the category of finite bipointed sets, dual to the category of primitive finite cubical sets. (By Morita equivalence it is not necessary to add “skeletal”.)

Definition 3 A **presheaf** is an object of a CDO.

In the language of category theory a presheaf is a functor $A : T \rightarrow \mathbf{Set}$ where T is the theory of the CDO.

(We don't *need* this larger language of category theory since the CDO framework does not depend on it, but it's convenient notationally, conceptually, and (higher) algebraically.

A complete CDO is a category of presheaves, namely

$$\mathbf{Set}^T = \mathbf{Set}^{J^{op}} = \mathbf{Psh}(J).$$

Yoneda embedding: J fully embeds in $\mathbf{Set}^{J^{op}}$. This is the sense in which the objects of J , or sorts, can be understood as primitive presheaves or unary algebras.

PART II. Chu Spaces, Untyped and Typed

In PART I homomorphisms transformed algebras covariantly.

We gave (hopefully) enough examples to illustrate the CDO approach.

The CDO framework can be taken much further, for example the notion and role of topos, higher arities than unary, coalgebras, etc.

The main goal of this talk is to show how the CDO framework generalizes easily to incorporate the open sets of point set topology, the functionals of linear algebra, etc. These transform contravariantly.

Distinguished objects are now of two kinds, sorts as in PART I and properties. Morphisms from sorts are elements as before, or **points**. Morphisms to properties are **states**.

This methodology was first employed in point set topology. It can also be usefully observed in linear algebra, and generalizes in a number of ways.

Chu spaces

Basic case: Chu spaces over $2 = \{0, 1\}$.

The base category is a category $\mathbf{1} \begin{smallmatrix} \xrightarrow{0} \\ \xleftarrow{1} \end{smallmatrix} \perp$.

A category \mathcal{C} of Chu spaces $\mathcal{A}, \mathcal{B}, \dots$ and their transformations is a category with rigid distinguished objects $\mathbf{1}$ and \perp and morphisms

$$\mathbf{1} \begin{smallmatrix} \xrightarrow{0} \\ \xleftarrow{1} \end{smallmatrix} \perp.$$

A **point** a is a morphism $a : \mathbf{1} \rightarrow \mathcal{A}$. Notation: $a \in A$.

A **state** x is a morphism $x : \mathcal{A} \rightarrow \perp$. Notation: $x \in X$.

Composition xa of a state x with a point a is 0 or 1. It defines an $A \times X$ matrix $c : A \times X \rightarrow \{0, 1\}$ of 0's and 1's, where \mathcal{A} is understood as a triple (A, c, X) .

Transformations of Chu spaces

A **transformation** $h : \mathcal{A} \rightarrow \mathcal{B}$ where $\mathcal{A} = (A, c, X)$, $\mathcal{B} = (B, d, Y)$ acts

- covariantly on a point $a \in A$ yielding a point $ha \in B$ (as in PART I);
- contravariantly on a state $y \in Y$ yielding a state $yh \in X$.

Picture: $\mathbf{1} \xrightarrow{a} \mathcal{A} \xrightarrow{h} \mathcal{B} \xrightarrow{y} \perp$. Here h can be interpreted as a pair (f, g) of functions $f : A \rightarrow B$, $g : Y \rightarrow X$ defined by the actions of h on a and y respectively.

By associativity $y(ha) = (yh)a$. This is read more conventionally as an adjointness condition

$$\forall a \in A \forall y \in Y. d(h(a), y) = c(a, g(y)).$$

In point set topology, the definition of continuity based on preservation of open sets under inverse image can be restated as such an adjointness requirement.

Requisite modifications to PART I

Mod 1. *Definition.* Two transformations $h, k : \mathcal{A} \rightarrow \mathcal{B}$ are **equivalent** when they have the same action on every point of \mathcal{A} and every state of \mathcal{B} .

Axiom A1 is then understood for this more general case.

Mod 2. Extensions must preserve both points and states of Chu spaces.

Everything else works in the same way.

The category **Chu**₂ of all Chu spaces over 2 and their transformations is the complete CDO on $\mathbf{1} \begin{matrix} \xrightarrow{0} \\ \xrightarrow{1} \end{matrix} \perp$.

Generalizing Chu2 to ChuK

A simple generalization is to replace $\mathbf{1} \begin{smallmatrix} 0 \\ \xrightarrow{\quad} \\ 1 \end{smallmatrix} \perp$ by $\mathbf{1} \xrightarrow{K} \perp$ where K is an arbitrary set whose elements supply the matrix entries or values of xa .

A complete CDO on such a base forms the category \mathbf{Chu}_K .

One of a number of master theorems about Chu categories:

Theorem

Every elementary class of total arity k and its relation-preserving homomorphisms fully embeds in \mathbf{Chu}_{2^k} .

More at <http://chu.stanford.edu>

Generalizing to typed Chu spaces

The \mathbf{Chu}_K base $\mathbf{1} \xrightarrow{K} \perp$ where K is an arbitrary set can be understood as two one-object categories $\mathbf{1}$ and \perp and a set K of morphisms from $\mathbf{1}$ to \perp .

Generalize $\mathbf{1}$ to a small category \mathcal{J} , \perp to a small category \mathcal{L} , and K to a doubly indexed family $K_{j\ell}$ of morphisms from $j \in \text{ob}(\mathcal{J})$ to $\ell \in \text{ob}(\mathcal{L})$.

Graph theorists call this a bipartite graph. Category theorists call $\mathcal{L} \nrightarrow \mathcal{J}$.

A typed Chu space consists of a presheaf on \mathcal{J} , a dual presheaf on \mathcal{L} , and what I shall call qualia $k : j \rightarrow \ell$ in $K_{j\ell}$.

PART III: Three Questions of Philosophy

- 1 In Descartes's mind-body dichotomy, is mind an equal partner to body, as opposed to just a convenient conceptual artifact?
- 2 Can we speak of the extension of a property like color or weight by analogy with the extension of a sort like cat or dog as the set of its members?
- 3 Are C.I. Lewis's qualia logically consistent?

Answers informed by (typed) Chu spaces.

- 1 Yes. Associate points and states with respectively individuals (concrete) and predicates (conceptual).
- 2 Yes. The extension of a sort (property) in a universe is the set of its individuals of that sort (states of that property).
- 3 Yes. Lewis conceived of qualia such as red or heavy as both concrete and psychological. Typed Chu spaces on $\mathcal{K} : \mathcal{L} \nrightarrow \mathcal{J}$ make a quale *calico* : *cat* \rightarrow *color* simultaneously a color-state of the primitive cat and the unique calico cat in the universe of colors.